Back-Paper ISI Bangalore June 2021

Differential geometry II 100 Points

Notes.
(a) The duration of this exam is three hours.
(b) You may freely use any result proved in class or in the text-book. Justify all other steps.
(c) $\mathbb{R}$ = real numbers, $\mathbb{R}P^n$ = real projective n-space, S^1 = the unit circle in $\mathbb{R}^2$.

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1. [20 points] Prove that the topological space $M = S^1 \times S^1$ is a C^∞ manifold by exhibiting a C^∞ atlas on it. Find an embedding of M in some $\mathbb{R}^n$ as a regular submanifold.

2. [20 points] Let $f\colon \mathbb{R}P^n \to \mathbb{R}^k$ be a C^∞ map. Prove that there is at least one point in the image over which f is not a submersion.

3. [20 points] Verify whether the common solution set of $x^n + y^n + z^n = 1$ and $x + y + z = 0$ is a regular submanifold of $\mathbb{R}^3$ or not.

4. [20 points] Find all the left-invariant vector fields on the additive Lie group $\mathbb{R}^n$ and the multiplicative Lie group S^1.

5. [10 points] Compute the Lie bracket $[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},\ e^y\frac{\partial}{\partial x}]$.

6. [10 points] For the map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(r,\theta) = (r\cos\theta,\, r\sin\theta)$ compute $F^*(dx \wedge dy)$ where x, y are the coordinate functions on the target space.